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On the solutions of second order generalized difference equations

M Maria Susai Manuel[1], Adem Kılıçman[2*], G Britto Antony Xavier[3], R Pugalarasu[3] and DS Dilip[3]

*Correspondence: akilicman@putra.upm.edu.my
[2]Department of Mathematics and Institute for Mathematical Research, University Putra Malaysia, Serdang, Selangor 43400, Malaysia
Full list of author information is available at the end of the article

Abstract

In this article, the authors discuss $\ell_{2(\ell)}$ and $c_{0(\ell)}$ solutions of the second order generalized difference equation

$$\Delta_\ell^2 u(k) + f(k, u(k)) = 0, \quad k \in [a, \infty), a > 0$$

and we prove the condition for non existence of non-trivial solution where $\Delta_\ell u(k) = u(k+\ell) - u(k)$ for $\ell > 0$. Further we present some formulae and examples to find the values of finite and infinite series in number theory as application of Δ_ℓ.
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1 Introduction

Difference equations usually describe the evolution of some certain phenomena over time and are also important in describing dynamics for fundamentally discrete system, see [1]. For example, in the numerical integration, the standard approach is to use the difference equations. Similarly, the population dynamics have discrete generations; the size of the $(k+1)$st generation $u(k+1)$ is a function of the kth generation $u(k)$. This can be expressed as difference equation of the form

$$u(k+1) = f\big(u(k)\big),$$

see for example [2]. Further, the concept of difference equations with many examples in applications such as asymptotic behavior of solutions of difference equations were studied extensively by Elaydi [3] where the analytic and geometric approaches were also combined in order to studying difference equations. Further, in [3], both classical and modern treatment of the difference equations were presented in excellent form. For related results on difference equations, see [4–8]. In the present article, we study $\ell_{2(\ell)}$ and $c_{0(\ell)}$ solutions of the following second order generalized difference equation

$$\Delta_\ell^2 u(k) + f\big(k, u(k)\big) = 0, \quad k \in [a, \infty), a > 0, \tag{1}$$

where $\Delta_\ell u(k) = u(k+\ell) - u(k)$ for $\ell > 0$. We provide some related definitions and development for the present article.

The basic theory of difference equations is based on the operator Δ defined as

$$\Delta u(k) = u(k+1) - u(k), \quad k \in \mathbb{N}, \tag{2}$$

where $\mathbb{N} = \{0, 1, 2, 3, \dots, \}$. Even though many authors [1–4] have suggested the definition of Δ as

$$\Delta u(k) = u(k+\ell) - u(k), \quad k \in \mathbb{N}, \ell \in \mathbb{R} - \{0\} \tag{3}$$

and there are several research took place on this line. By defining Δ_ℓ and its inverse Δ_ℓ^{-1}, many interesting results and applications in number theory as well as in fluid dynamics can be obtained. By extending the study for sequences of complex numbers and ℓ to be real, some new qualitative properties like rotatory, expanding, shrinking, spiral and weblike structures were studied for the solutions of difference equations involving Δ_ℓ. For similar results, we refer to [9–13].

In particular, the ℓ_2 and c_0 solutions of second order difference equations of (1) when $\ell = 1$, were discussed in [8]. In this article, we discuss $\ell_{2(\ell)}$ and $c_{0(\ell)}$ solutions for the second order generalized difference Equation (1) and present some applications of Δ_ℓ in the finite and infinite series of number theory. Throughout this article, we use the following notation:

(i) $[k]$ denotes the integer part of k,
(ii) $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, $\mathbb{N}(a) = \{a, a+1, a+2, \dots\}$,
(iii) $\mathbb{N}_\ell(j) = \{j, j+\ell, j+2\ell, \dots\}$ and $\mathbb{R}$ is the set of all real numbers.

2 Preliminaries

In this section, we present some of the preliminary definitions and related results which will be useful for future discussion. The following three definitions held in [9].

Definition 2.1 Let $u : [0, \infty) \to \mathbb{C}$ and $\ell \in (0, \infty)$ then, the generalized difference operator Δ_ℓ is defined as

$$\Delta_\ell u(k) = u(k+\ell) - u(k). \tag{4}$$

Similarly, the generalized difference operator of the rth kind is defined as

$$\Delta_\ell^r = \Delta_\ell\left(\Delta_\ell^{r-1}\right) \quad \text{if } r \geq 2. \tag{5}$$

Definition 2.2 For arbitrary $x, y \in \mathbb{R}$ the h-factorial function is defined by

$$x_h^{(y)} = h^y \frac{\Gamma(\frac{x}{h}+1)}{\Gamma(\frac{x}{h}+1-y)}, \tag{6}$$

where Γ is the Euler gamma function. Note that when $x = k$, $h = \ell$, $y = n \in \mathbb{N}(1)$ Definition 2.2 coincides with Definition 2.1.

Definition 2.3 Let $u(k)$, $k \in [0, \infty)$ be a real or complex valued function and $\ell \in (0, \infty)$. Then, the inverse of Δ_ℓ denoted by Δ_ℓ^{-1} and defined as follows.

$$\text{If } \Delta_\ell v(k) = u(k), \text{ then } v(k) = \Delta_\ell^{-1} u(k) + c_j, \tag{7}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - \lceil \frac{k}{\ell} \rceil \ell$.

Definition 2.4 The generalized polynomial factorial for $\ell > 0$ is defined as

$$k_\ell^{(n)} = k(k-\ell)(k-2\ell)\cdots\big(k-(n-1)\ell\big). \tag{8}$$

Lemma 2.5 *If $\ell > 0$ and $n \in \mathbb{N}_\ell(1)$ then,*

$$\Delta_\ell^{-1} k_\ell^{(n)} = \frac{1}{(n+1)\ell}(k-\ell)_\ell^{(n+1)} + c_j \tag{9}$$

for all $k \in \mathbb{N}_\ell(j)$, $j = k - \lceil \frac{k}{\ell} \rceil \ell$ and c_j is constant.

Lemma 2.6 ([13] Product formula) *Let $u(k)$ and $v(k)$ be any two functions. Then*

$$\begin{aligned}\Delta_\ell\big\{u(k)v(k)\big\} &= u(k+\ell)\Delta_\ell v(k) + v(k)\Delta_\ell u(k)\\ &= v(k+\ell)\Delta_\ell u(k) + u(k)\Delta_\ell v(k), \quad \forall k \in \mathbb{N}_\ell(a).\end{aligned} \tag{10}$$

Lemma 2.7 ([12]) *Let $\ell > 0$, $n \in \mathbb{N}(2)$, $k \in (\ell, \infty)$ and $k_\ell^{(n)} \neq 0$. Then,*

$$\Delta_\ell^{-1} \frac{1}{k_\ell^{(n)}} = \frac{-1}{(n-1)\ell(k-\ell)_\ell^{(n-1)}} + c_j. \tag{11}$$

Definition 2.8 A function $u(k)$, $k \in [a, \infty)$ is said to be in the space $\ell_{2(\ell)}$, if

$$\sum_{\gamma=0}^{\infty} \big|u(a+j+\gamma\ell)\big|^2 < \infty \quad \text{for all } j \in [0, \ell). \tag{12}$$

If $\lim_{r\to\infty} |u(a+j+r\ell)| = 0$, for all $0 \le j < \ell$ then $u(k)$ is said to be in the space $c_{0(\ell)}$.

Lemma 2.9 ([9] Summation formula of finite series) *If real valued function $u(k)$ is defined for all $k \in [0, \infty)$, then*

$$\Delta_\ell^{-1} u(k) = \sum_{r=1}^{\lceil \frac{k}{\ell} \rceil} u(k - r\ell) + c_j, \tag{13}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - \lceil \frac{k}{\ell} \rceil \ell$. Since $[0,\infty) = \bigcup_{0 \le j < \ell} N_\ell(j)$, each complex number c_j, ($0 \le j < \ell$) is called an initial value of $k \in N_\ell(j)$. Usually, each initial value c_j is taken from any one of the values $u(j)$, $u(j+\ell)$, $u(j+2\ell)$, etc.

Lemma 2.10 (Summation formula of infinite series) *If $\lim_{k\to\infty} u(k) = 0$ and $\ell > 0$, then*

$$\Delta_\ell^{-1} u(k) = -\sum_{r=0}^{\infty} u(k + r\ell). \tag{14}$$

Proof Assume $z(k) = \sum_{r=0}^{\infty} u(k + r\ell)$. Then,

$$\Delta_\ell z(k) = \sum_{r=0}^{\infty} u(k + \ell + r\ell) - \sum_{r=0}^{\infty} u(k + r\ell) = -u(k).$$

Now, the proof follows from $\lim_{k\to\infty} u(k) = 0$ and Definition 2.3. □

Theorem 2.11 *If* $\lim_{k\to\infty} u(k) = 0$ *and* $\ell > 0$, *then*

$$\Delta_\ell^{-2} u(k) = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} u(k + r_1\ell + r_2\ell). \tag{15}$$

Proof The proof follows by taking Δ_ℓ^{-1} on (14). □

Corollary 2.12 *Let* $k \in [\ell, \infty)$ *and* $\ell \in (0, \infty)$. *Then*

$$\Delta_\ell^{-1} \frac{1}{k(k-\ell)} = -\frac{1}{\ell(k-\ell)}$$

and hence

$$\sum_{r=0}^{\infty} \frac{1}{(k + r\ell)(k + r\ell - \ell)} = \frac{1}{\ell(k-\ell)}. \tag{16}$$

Proof The proof follows from Equation (14) and $c_j = 0$ as $k \to \infty$. □

The following example illustrates Corollary 2.12.

Example 2.13 Taking $\ell = 0.8$, $k = 1$ in (16), we obtain

$$\frac{1}{1 \times 0.2} + \frac{1}{1.8 \times 1} + \frac{1}{2.6 \times 1.8} + \cdots = \frac{1}{0.8 \times 0.2}.$$

The following example shows that $\frac{1}{k_\ell^{(n)}} \in c_{0(\ell)}$ and $\ell_{2(\ell)}$.

Example 2.14 Assume $n \in \mathbb{N}(2)$ and $k \in [n\ell, \infty)$. Let $u(k) = \frac{1}{k_\ell^{(n)}}$. By Lemmas 2.7 and 2.10, we obtain

$$\frac{1}{(n-1)\ell k_\ell^{(n-1)}} = \sum_{r=0}^{\infty} \frac{1}{(k + r\ell)_\ell^{(n)}}.$$

Since $c_j = 0$ as $k \to \infty$. Replacing k by $a + j$, we get

$$\sum_{r=0}^{\infty} \frac{1}{(a + j + r\ell)_\ell^{(n)}} = \frac{1}{(n-1)\ell(a+j)_\ell^{(n-1)}}, \quad \text{for } a \geq n\ell. \tag{17}$$

Since

$$\left|\frac{1}{(a + j + r\ell)_\ell^{(n)}}\right|^2 < \frac{1}{(a + j + r\ell)_\ell^{(n)}},$$

for $a \geq n\ell$ thus Equation (17) yields

$$\sum_{r=0}^{\infty} \left| u(a+j+r\ell) \right|^2 < \sum_{r=0}^{\infty} \frac{1}{(a+j+r\ell)_{\ell}^{(n)}} = \frac{1}{(n-1)\ell(a+j)^{(n-1)}} < \infty.$$

By Definition 2.8, the function $\frac{1}{k_{\ell}^{(n)}} \in \ell_{2(\ell)}$. Since

$$\lim_{r\to\infty} \frac{1}{(a+j+r\ell)_{\ell}^{(n)}} = 0, \qquad \frac{1}{k_{\ell}^{(n)}} \in c_{0(\ell)}.$$

Now taking $a = n\ell$ then $u(k)$ is an $\ell_{2(\ell)}$ space function.

3 Main results

In this section, we present the condition for non existence of non-trivial solution of (1).

Lemma 3.1 *Let $a \geq 2\ell$ and $k \in [a, \infty)$. Then*

$$\frac{1}{k} < \frac{4}{(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k}+\sqrt{k-\ell})}.$$

Proof We have

$$\begin{aligned}
&\frac{4}{(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k}+\sqrt{k-\ell})} \\
&\quad = \frac{4(\sqrt{k+\ell}-\sqrt{k})(\sqrt{k}-\sqrt{k-\ell})}{\ell^2} \\
&\quad = \frac{4}{\ell^2}\sqrt{k}\sqrt{k}\left[\left(1+\frac{\ell}{k}\right)^{\frac{1}{2}}-1\right]\left[1-\left(1-\frac{\ell}{k}\right)^{\frac{1}{2}}\right] \\
&\quad = \frac{4k}{\ell^2}\left[1+\frac{1}{2}\frac{\ell}{k}-\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3-\frac{1}{4!}\frac{1}{4}\frac{3}{2}\frac{5}{2}\left(\frac{\ell}{k}\right)^4+\cdots\right] \\
&\qquad \times\left[1-\left(1-\frac{1}{2}\frac{\ell}{k}-\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2-\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3-\frac{1}{4!}\frac{1}{4}\frac{3}{2}\frac{5}{2}\left(\frac{\ell}{k}\right)^4-\cdots\right)\right].
\end{aligned}$$

Since each positive term is greater than the consecutive negative term in the first expression, we find

$$\begin{aligned}
&\frac{4k}{\ell^2}\left[\frac{1}{2}\frac{\ell}{k}-\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2\right]\times\left[\frac{1}{2}\frac{\ell}{k}+\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3+\frac{1}{4!}\frac{1}{4}\frac{3}{2}\frac{5}{2}\left(\frac{\ell}{k}\right)^4+\cdots\right] \\
&\quad = \frac{4}{\ell^2}\left[\frac{\ell}{2}-\frac{\ell}{2}\frac{1}{4}\frac{\ell}{k}\right]\left[\frac{1}{2}\frac{\ell}{k}+\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3+\frac{1}{4!}\frac{1}{4}\frac{3}{2}\frac{5}{2}\left(\frac{\ell}{k}\right)^4+\cdots\right] \\
&\quad = \frac{4}{\ell^2}\frac{\ell}{2}\left[\frac{1}{2}\frac{\ell}{k}+\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3+\frac{1}{4!}\frac{1}{4}\frac{3}{2}\frac{5}{2}\left(\frac{\ell}{k}\right)^4+\cdots\right] \\
&\qquad -\frac{4}{\ell^2}\frac{\ell}{2}\frac{1}{4}\frac{\ell}{k}\left[\frac{1}{2}\frac{\ell}{k}+\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3+\frac{1}{4!}\frac{1}{4}\frac{3}{2}\frac{5}{2}\left(\frac{\ell}{k}\right)^4+\cdots\right] \\
&\quad = \frac{1}{k}+\frac{2}{\ell}\left[\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2+\frac{1}{3!}\frac{1}{4}\frac{3}{2}\left(\frac{\ell}{k}\right)^3+\frac{1}{4!}\frac{1}{4}\frac{3}{2}\frac{5}{2}\left(\frac{\ell}{k}\right)^4+\cdots\right]
\end{aligned}$$

$$- \frac{2}{\ell}\left[\frac{1}{2!}\frac{1}{4}\left(\frac{\ell}{k}\right)^2 + \frac{1}{2!}\frac{1}{4}\frac{1}{4}\left(\frac{\ell}{k}\right)^3 + \frac{1}{3!}\frac{1}{4}\frac{1}{4}\frac{1}{4}\left(\frac{\ell}{k}\right)^4 + \cdots\right]$$
$$= \frac{1}{k} + \frac{2}{4\ell}\left[\frac{1}{3!}\left(\frac{3}{2} - \frac{3}{4}\right)\left(\frac{\ell}{k}\right)^3 + \frac{1}{4!}\frac{3}{2}\left(\frac{5}{2} - \frac{4}{4}\right)\left(\frac{\ell}{k}\right)^4\right.$$
$$\left. + \frac{1}{5!}\frac{3}{2}\frac{5}{2}\left(\frac{7}{2} - \frac{5}{4}\right)\left(\frac{\ell}{k}\right)^5 + \frac{1}{6!}\frac{3}{2}\frac{5}{2}\frac{7}{2}\left(\frac{9}{2} - \frac{6}{4}\right)\left(\frac{\ell}{k}\right)^6 + \cdots\right] > \frac{1}{k},$$

since the second term is positive. □

Lemma 3.2 *Let $a \geq 2\ell$ and $k \in [a, \infty)$. Then*

$$\frac{\sqrt{k+\ell}}{\sqrt{k}} - \frac{\sqrt{k}}{\sqrt{k+\ell} + \sqrt{k-\ell}} < 1. \tag{18}$$

Proof From the Binomial theorem for rational index, we find

$$\begin{aligned}
\frac{\sqrt{k+\ell}}{\sqrt{k}} - \frac{\sqrt{k}}{\sqrt{k+\ell} + \sqrt{k-\ell}} &= \left(1 + \frac{\ell}{k}\right)^{\frac{1}{2}} - \frac{\sqrt{k}}{2\ell}\left[(k+\ell)^{\frac{1}{2}} - (k-\ell)^{\frac{1}{2}}\right] \\
&= 1 + \frac{1}{2}\frac{\ell}{k} - \frac{1}{2!}\frac{1}{2}\frac{1}{2}\left(\frac{\ell}{k}\right)^2 + \frac{1}{3!}\frac{1}{2}\frac{1}{2}\frac{3}{2}\left(\frac{\ell}{k}\right)^3 - \cdots \\
&\quad - \frac{k}{2\ell}\left[1 + \frac{1}{2}\frac{\ell}{k} - \frac{1}{2!}\frac{1}{2}\frac{1}{2}\left(\frac{\ell}{k}\right)^2 + \frac{1}{3!}\frac{1}{2}\frac{1}{2}\frac{3}{2}\left(\frac{\ell}{k}\right)^3 - \cdots\right. \\
&\quad \left. - \left(1 - \frac{1}{2}\frac{\ell}{k} - \frac{1}{2!}\frac{1}{2}\frac{1}{2}\left(\frac{\ell}{k}\right)^2 - \frac{1}{3!}\frac{1}{2}\frac{1}{2}\frac{3}{2}\left(\frac{\ell}{k}\right)^3 - \cdots\right)\right] \\
&= 1 + \frac{1}{2}\frac{\ell}{k} - \frac{1}{2!}\frac{1}{2}\frac{1}{2}\left(\frac{\ell}{k}\right)^2 + \frac{1}{3!}\frac{1}{2}\frac{1}{2}\frac{3}{2}\left(\frac{\ell}{k}\right)^3 - \cdots \\
&\quad - \frac{k}{2\ell}\left[\frac{\ell}{k} + \frac{1}{3!}\frac{1}{2}\frac{1}{2}\frac{3}{2}\left(\frac{\ell}{k}\right)^3 + \cdots\right].
\end{aligned}$$

Since each negative terms is greater than the next consecutive positive term and $k \geq 2\ell$, we get

$$\frac{\sqrt{k+\ell}}{\sqrt{k}} - \frac{\sqrt{k}}{\sqrt{k+\ell} + \sqrt{k-\ell}} = 1 + \frac{1}{2}\frac{\ell}{k} - \frac{1}{2} = \frac{1}{2} + \frac{1}{2}\frac{\ell}{k} < 1.$$ □

Lemma 3.3 *Let $a \geq 2\ell$. If*

$$\Delta_\ell z(k) \leq \alpha(k) + \beta(k)z(k) \tag{19}$$

and $\frac{-\ell}{k} < \beta < \frac{-\ell^2}{k^2}$ for all $k \in [a, \infty)$ then

$$\Delta_\ell\left(z(k)\prod_{r=0}^{\lceil\frac{k-a}{\ell}\rceil-1}\left(1 + \beta(j + a + r\ell)\right)^{-1}\right) \leq \alpha(k)\prod_{r=0}^{\lceil\frac{k-a}{\ell}\rceil}\left(1 + \beta(j + a + r\ell)\right)^{-1}, \tag{20}$$

where $j = k - a - \lceil\frac{k-a}{\ell}\rceil\ell$.

Proof From the inequality (19) and $1 + \beta(k) > 0$ for all $k \in [a, \ell)$, we find,

$$\frac{z(k+\ell)}{1+\beta(k)} - z(k) \leq \frac{\alpha(k)}{1+\beta(k)}$$

which yields,

$$\frac{z(k+\ell)}{1+\beta(k)} \prod_{r=0}^{\lceil\frac{k-a}{\ell}\rceil-1} \bigl(1+\beta(j+a+r\ell)\bigr)^{-1} - z(k) \prod_{r=0}^{\lceil\frac{k-a}{\ell}\rceil-1} \bigl(1+\beta(j+a+r\ell)\bigr)^{-1}$$
$$\leq \frac{\alpha(k)}{1+\beta(k)} \prod_{r=0}^{\lceil\frac{k-a}{\ell}\rceil-1} \bigl(1+\beta(j+a+r\ell)\bigr)^{-1}.$$

Now (20) follows by taking $r = \lceil\frac{k-a}{\ell}\rceil$ and $j + a + \lceil\frac{k-a}{\ell}\rceil\ell = k$. □

The following theorem shows the nonexistence of solutions of (3).

Theorem 3.4 *For all $(k, u) \in [a, \infty) \times \mathbb{R}$, let the function $f(k, u)$ be defined and*

$$\bigl|f(k,u)\bigr| \leq \frac{\ell^2}{2} k^{-2} |u|. \tag{21}$$

Then, if $u(k) \in \ell_{2(\ell)}$ is a solution of (3), there exists a real $k_1 \geq a$ ($a \geq 2\ell$) such that $u(k) = 0$ for all $k \in [k_1, \infty)$.

Proof Since $u(k)$ is a solution of (3) and belong to $\ell_{2(\ell)}$, we have $\sum_{r=0}^{\infty} |u(a + j + r\ell)|^2 < \infty$ which yields $\lim_{k\to\infty} u(k) = 0$ and hence

$$\lim_{k\to\infty} \Delta_\ell u(k) = \lim_{k\to\infty} \Delta_\ell^2 u(k) = 0. \tag{22}$$

By using Equations (3) and (22), and applying Δ_ℓ^{-1} on Equation (3) with Lemma 2.10, we obtain

$$\Delta_\ell u(k) = \sum_{r=0}^{\infty} f\bigl(k + r\ell, u(k + r\ell)\bigr). \tag{23}$$

Now by applying again Δ_ℓ^{-1} on both sides, and by Theorem 2.10, we get

$$u(k) = -\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} f\bigl(k + r\ell + s\ell, u(k + r\ell + s\ell)\bigr) \tag{24}$$

which yields

$$u(k) = -\sum_{r=0}^{\infty} (r+1) f\bigl(k + r\ell, u(k + r\ell)\bigr), \quad k \in [a, \infty). \tag{25}$$

Therefore, from (21), we obtain

$$\bigl|u(k)\bigr| \leq \frac{\ell^2}{2} \nu(k), \tag{26}$$

where

$$v(k) = \sum_{r=0}^{\infty}(r+1)(k+r\ell)^{-2}\left|u(k+r\ell)\right|, \quad \text{for all } k \in [a,\infty). \tag{27}$$

Obviously $v(k) \geq 0$ for all $k \in [a,\infty)$ and $\lim_{k\to\infty} v(k) = 0$.

If $v(k+j) = 0$, $\forall j \in [0,\ell)$, for some $k = k_1 \geq a$, then $(r+1)(k+j+r\ell)^{-2}u(k+j+r\ell) = 0$ for all $r = 0,1,2,\ldots$. Hence $u(k) = 0$ for all $k \geq k_1$. In this case, the proof is complete.

Now, we suppose that $v(k) > 0$ for all $k \in [a,\infty)$, from (27), we have

$$\Delta_\ell v(k) = -\sum_{r=0}^{\infty}(k+r\ell)^{-2}\left|u(k+r\ell)\right|$$

and

$$\Delta_\ell^2 v(k) = k^{-2}\left|u(k)\right|.$$

From (26), we have

$$\Delta_\ell^2 v(k) \leq \frac{\ell^2}{2}k^{-2}v(k) \quad \text{for all } k \in [a,\infty). \tag{28}$$

From (27), $a \geq 2\ell$, $\frac{r+1}{k+r\ell} \leq \frac{1}{\ell}$, by Schwartz's inequality, we obtain

$$v(k) \leq \ell^{-1}\sum_{r=0}^{\infty}(k+r\ell)^{-1}\left|u(k+r\ell)\right| \leq \ell^{-1}\left(\sum_{r=0}^{\infty}(k+r\ell)^{-2}\right)^{\frac{1}{2}}\left(\sum_{r=0}^{\infty}\left|u(k+r\ell)\right|^2\right)^{\frac{1}{2}}.$$

By using Corollary 2.12, we get

$$v(k) \leq \ell^{-\frac{3}{2}}\frac{1}{\sqrt{k-\ell}}\left(\sum_{r=0}^{\infty}\left|u(k+r\ell)\right|^2\right)^{\frac{1}{2}}.$$

If $w(k) = \ell^{\frac{3}{2}}\sqrt{k-\ell}\,v(k)$, then

$$w(k) \leq \left(\sum_{r=0}^{\infty}\left\|u(k+r\ell)\right\|^2\right)^{\frac{1}{2}}, \quad \text{for all } k \in [a,\infty). \tag{29}$$

Hence we have

$$w(k) \to 0 \quad \text{and} \quad w(k) > 0, \quad \forall k \in [a,\infty). \tag{30}$$

By applying Lemma 2.6 to Equation (29) twice, we obtain

$$\Delta_\ell^2 w(k) = \ell^{\frac{3}{2}}\left(\sqrt{k+\ell}\,\Delta_\ell^2 v(k) + 2\Delta_\ell v(k)\Delta_\ell\sqrt{k} + v(k)\Delta_\ell^2\sqrt{k-\ell}\right). \tag{31}$$

Again from Lemma 2.6 and Equation (29), we obtain

$$\Delta_\ell v(k) = \ell^{-\frac{3}{2}}\left(\frac{1}{\sqrt{k}}\Delta_\ell w(k) + w(k)\Delta_\ell\frac{1}{\sqrt{k-\ell}}\right). \tag{32}$$

From (31), (32) and by Lemma 2.6, we find that

$$\begin{aligned}
&\Delta_\ell\left(\frac{1}{k-\ell}\Delta_\ell w(k)\right)\\
&= \frac{1}{k}\Delta_\ell^2 w(k) - \left(\frac{\ell}{k(k-\ell)}\right)\Delta_\ell w(k)\\
&= \frac{\ell^{\frac{3}{2}}}{k}\left\{\sqrt{k+\ell}\Delta_\ell^2 v(k) + 2\Delta_\ell v(k)\Delta_\ell\sqrt{k} + v(k)\Delta_\ell^2\sqrt{k-\ell}\right\}\\
&\quad - \left(\frac{\ell}{k(k-\ell)}\right)\Delta_\ell w(k)\\
&= \frac{\ell^{\frac{3}{2}}}{k}\left\{\sqrt{k+\ell}\Delta_\ell^2 v(k) + 2\ell^{-\frac{3}{2}}\left[\frac{1}{\sqrt{k}}\Delta_\ell w(k) + 2\frac{w(k)}{k}\Delta_\ell\frac{1}{\sqrt{k-\ell}}\right]\Delta_\ell\sqrt{k}\right.\\
&\quad \left. + \frac{\ell^{\frac{3}{2}}}{k}v(k)\Delta_\ell^2\sqrt{k-\ell}\right\} - \left(\frac{\ell}{k(k-\ell)}\right)\Delta_\ell w(k)\\
&= \ell^{\frac{3}{2}}\left(\frac{\sqrt{k+\ell}}{k}\right)\Delta_\ell^2 v(k) + \frac{2}{k}\ell^{\frac{3}{2}}\sqrt{k-\ell}\,v(k)\Delta_\ell\frac{1}{\sqrt{k-\ell}}\Delta_\ell\sqrt{k}\\
&\quad + \frac{\ell^{\frac{3}{2}}}{k}v(k)\Delta_\ell^2\sqrt{k-\ell} + \frac{2}{k\sqrt{k}}\Delta_\ell w(k)\Delta_\ell\sqrt{k} - \frac{\ell}{k(k-\ell)}\Delta_\ell w(k)\\
&\le \ell^{\frac{3}{2}}\left(\frac{\ell^2\sqrt{k+\ell}}{2k^3}\right)v(k) + \frac{2\ell^{\frac{3}{2}}}{k}\sqrt{k-\ell}\,v(k)\Delta_\ell\sqrt{k}\Delta_\ell\frac{1}{\sqrt{k-\ell}}\\
&\quad + \frac{\ell^{\frac{3}{2}}}{k}v(k)\Delta_\ell^2\sqrt{k-\ell}\\
&\quad + \left(\frac{2(k-\ell)}{k\sqrt{k}}\Delta_\ell\sqrt{k} - \frac{\ell}{k}\right)\frac{1}{k-\ell}\Delta_\ell w(k)
\end{aligned}$$

which in view of (28), (30) gives

$$\Delta_\ell z(k) \le \alpha(k) + \beta(k)z(k), \tag{33}$$

where

$$z(k) = \frac{1}{k-\ell}\Delta_\ell w(k), \tag{34}$$

$$\alpha(k) = \ell^{\frac{3}{2}}\left(\frac{\ell^2\sqrt{k+\ell}}{2k^3} + \frac{2}{k}\sqrt{k-\ell}\Delta_\ell\sqrt{k}\Delta_\ell\frac{1}{\sqrt{k-\ell}} + \frac{1}{k}\Delta_\ell^2\sqrt{k-\ell}\right)v(k) \tag{35}$$

and

$$\beta(k) = \left(\frac{2(k-\ell)}{k\sqrt{k}}\right)\Delta_\ell\sqrt{k} - \frac{\ell}{k}. \tag{36}$$

Since $(\frac{2(k-\ell)}{k\sqrt{k}})\Delta_\ell\sqrt{k} > 0$, from $(1+\frac{\ell}{k})^{\frac{1}{2}} < 1 + \frac{1}{2}\frac{\ell}{k}$, we obtain

$$-\frac{\ell}{k} < \beta(k) < -\frac{\ell^2}{k^2}, \quad \text{where } k \in [a,\infty). \tag{37}$$

Further, since

$$\begin{aligned}\sqrt{k}\sqrt{k-\ell}\Delta_\ell\sqrt{k}\Delta_\ell\frac{1}{\sqrt{k-\ell}} &= (\sqrt{k+\ell}-\sqrt{k})(\sqrt{k-\ell}-\sqrt{k})\\ &= -\frac{\ell^2}{(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k-\ell}+\sqrt{k})}\end{aligned}$$

and

$$\begin{aligned}\Delta_\ell^2\sqrt{k-\ell} &= \sqrt{k+\ell}-\sqrt{k}+\sqrt{k-\ell}-\sqrt{k}\\ &= \frac{(\sqrt{k+\ell}-\sqrt{k})(\sqrt{k+\ell}+\sqrt{k})}{(\sqrt{k+\ell}+\sqrt{k})}+\frac{(\sqrt{k-\ell}-\sqrt{k})(\sqrt{k-\ell}+\sqrt{k})}{(\sqrt{k-\ell}+\sqrt{k})}\\ &= \ell\frac{\sqrt{k-\ell}-\sqrt{k+\ell}}{(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k-\ell}+\sqrt{k})}\\ \gamma(k) &= \frac{\ell^{\frac{3}{2}}}{k\sqrt{k}}\left(\frac{\ell^2\sqrt{k+\ell}}{2k\sqrt{k}}+\frac{-2\ell^2+\ell\sqrt{k}(\sqrt{k-\ell}-\sqrt{k+\ell})}{(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k}+\sqrt{k-\ell})}\right)\nu(k).\end{aligned}$$

From Lemmas 3.1 and 3.2

$$\begin{aligned}\gamma(k) &< \frac{\ell^{\frac{3}{2}}}{k\sqrt{k}}\left(\frac{\ell^2\sqrt{k+\ell}}{2\sqrt{k}}\frac{4}{(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k}+\sqrt{k-\ell})}\right.\\ &\quad\left.+\frac{-2\ell^2+\ell\sqrt{k}(\sqrt{k-\ell}-\sqrt{k+\ell})}{(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k}+\sqrt{k-\ell})}\right)\nu(k)\\ &= \frac{2\ell^{\frac{3}{2}}}{k\sqrt{k}(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k}+\sqrt{k-\ell})}\left(\frac{\ell^2\sqrt{k+\ell}}{\sqrt{k}}-\frac{\ell^2\sqrt{k}}{\sqrt{k+\ell}+\sqrt{k-\ell}}-\ell^2\right)\nu(k)\\ &= \frac{2\ell^{\frac{7}{2}}}{k\sqrt{k}(\sqrt{k+\ell}+\sqrt{k})(\sqrt{k}+\sqrt{k-\ell})}\left(\frac{\sqrt{k+\ell}}{\sqrt{k}}-\frac{\sqrt{k}}{\sqrt{k+\ell}+\sqrt{k-\ell}}-1\right)\nu(k). \qquad (38)\end{aligned}$$

By Lemma 3.2, we find $\gamma(k) < 0$ for all $k \in [a,\infty)$. Thus from Lemma 3.3 and $\gamma(k) < 0$, we find

$$\Delta_\ell\left(z(k)\prod_{r=0}^{\lceil\frac{k-a}{\ell}\rceil-1}\big(1+\beta(j+a+r\ell)\big)^{-1}\right) < 0, \quad \text{for all } k \in [a+\ell,\infty).$$

That is,

$$z(k)\prod_{r=0}^{\lceil\frac{k-a}{\ell}\rceil-1}\big(1+\beta(j+a+r\ell)\big)^{-1}$$

is decreasing by ℓ steps.

If

$$z(k)\prod_{r=0}^{\lceil\frac{k-a}{\ell}\rceil-1}\big(1+\beta(j+a+r\ell)\big)^{-1} > 0$$

for all $k \in [a+\ell,\infty)$, then $z(k) > 0$. From (34) we find $\Delta_\ell w(k) > 0$ and hence $w(k)$ is increasing by ℓ steps, but this contradicts (30).

If there exists a real $K \geq a+\ell$ such that

$$z(K)\prod_{r=0}^{\lceil\frac{K-a}{\ell}\rceil-1}\bigl(1+\beta(j+a+r\ell)\bigr)^{-1} = p_j < 0$$

for all $0 \leq j < \ell$, then

$$z(k)\prod_{r=0}^{\lceil\frac{k-a}{\ell}\rceil-1}\bigl(1+\beta(j+a+r\ell)\bigr)^{-1} < p_j$$

for all $k \in [K,\infty)$, that is,

$$z(k) < p_j\prod_{r=0}^{\lceil\frac{k-a}{\ell}\rceil-1}\bigl(1+\beta(j+a+r\ell)\bigr).$$

However from (37), since $1+\beta(k) > (k-\ell)/k > 0$ and $j = k - \lceil\frac{k-a}{\ell}\rceil\ell$, it follows that $z(k) < p_j(j+a-\ell)/(k-\ell)$, and hence from (34), we find $\Delta_\ell w(k) < p_j(j+a-\ell)$. Further, since

$$w(k) \to 0, \quad k \geq K+2\ell \quad \Rightarrow \quad \frac{1}{\ell}(k-K-\ell) \geq 1$$

we get $w(k+\ell) < w(k) + p_j(j+a-\ell)$ which yields $w(k) < w(k-\ell) + p_j(j+a-\ell)$ and hence we get

$$w(k) < w(K+\ell) + \frac{1}{\ell}p_j(j+a-\ell)(k-K-\ell)$$

for all $k \in [K+2\ell,\infty)$, since

$$k \geq K+2\ell \quad \Rightarrow \quad k-K \geq 2\ell, \qquad \frac{1}{\ell}(k-K-\ell) \geq 1.$$

But this implies that $w(k) \to -\infty$, and again we get a contradiction to (30).

Thus combining the above arguments, we conclude that our assumption $v(k) > 0$ for all $k \in [a,\infty)$ is not correct, and this completes the proof. □

Theorem 3.5 *For all* $(k,u) \in [0,\infty)\times\mathbb{R}$, *let the function* $f(k,u)$ *be defined and*

$$\bigl|f(k,u)\bigr| \leq \ell^q k^{-q}|u|, \quad q > \frac{5}{2}. \tag{39}$$

Then, if $u(k)$ *is a solution of (3)* $\in c_{0(\ell)}$, *there exists an integer* $k_1 \geq a$ $(a \geq 4\ell)$ *such that* $u(k) = 0$ *for all* $k \in [k_1,\infty)$.

Proof Let $u(k)$ be a solution of (3) such that $\lim_{k\to\infty}|u(k)| = 0$. Then,

$$\lim_{k\to\infty}\Delta_\ell u(k) = \lim_{k\to\infty}\Delta_\ell^2 u(k) = 0$$

for all $\ell > 0$. Thus, for this solution also the relation (24) holds. Further, since there exists a constant $c_j > 0$ such that $|u(k)| \le c_j$ for all $k \in [a, \infty)$, where $0 \le j = k - \lceil \frac{k}{\ell} \rceil \ell < \ell$, we find that

$$\begin{aligned}
\sum_{r=0}^{\infty}(r+1)\big|f\big((k+r\ell), u(k+r\ell)\big)\big| &\le \sum_{r=0}^{\infty}\left(r + \frac{k}{\ell}\ell^{q}(k+r\ell)^{-q}\big|u(k+r\ell)\big|\right) \\
&= \sum_{r=0}^{\infty}(k+r\ell)^{1-q}\ell^{q-1}\big|u(k+r\ell)\big| \\
&\le c_j\ell^{q-1}\sum_{r=0}^{\infty}(k+r\ell)^{1-q} \quad \text{where } j = k - \left\lceil \frac{k}{\ell} \right\rceil \ell \\
&= c_j\ell^{q-1}\left[k^{1-q} + \sum_{r=1}^{\infty}(k+r\ell)^{1-q}\right] \\
&= c_j\ell^{q-1}\left[k^{1-q} + \ell^{1-q}\sum_{r=1}^{\infty}\left(\frac{k}{\ell}+r\right)^{1-q}\right] \\
&= c_j\ell^{q-1}\left[k^{1-q} + \ell^{1-q}\left[\frac{(\frac{k}{\ell})^{2-q}}{2-q} + r\right]_{\frac{k}{\ell}}^{\infty}\right] \\
&= c_j\ell^{q-1}\left[k^{1-q} + \left(\frac{k^{2-q}}{\ell(q-2)}\right)\right] < \infty,
\end{aligned}$$

for all $k \in [k_1, \infty)$. Therefore, this solution also has the representation (24).

Now as in Theorem 3.4, we define

$$\bar{v}(k) = \sum_{r=0}^{\infty}(r+1)(k+r\ell)^{-q}\big|u(k+r\ell)\big| = \sum_{r=0}^{\infty}\ell^{-q}(r+1)\left(\frac{k}{\ell}+r\right)^{-q}\big|u(k+r\ell)\big|.$$

Since $q > \frac{5}{2}$, we find

$$\bar{v}(k) \le \ell^{-q}\sum_{r=0}^{\infty}(r+1)\left(\frac{k}{\ell}+r\right)^{-2}\big|u(k+r\ell)\big| = \ell^{2-q}\sum_{r=0}^{\infty}(r+1)(k+r)^{-2}\big|u(k+r\ell)\big|$$

then it follows that

$$\bar{v}(k) \le \ell^{2-q}\left(\frac{\ell^{-\frac{3}{2}}}{\sqrt{k-\ell}}\right)\left\{\sum_{r=0}^{\infty}\big|u(k+r\ell)\big|^2\right\}^{\frac{1}{2}}.$$

Hence we define

$$\begin{aligned}
\bar{w}(k) &= \ell^{q-\frac{1}{2}}\sqrt{k-\ell}\,\bar{v}(k), \\
\bar{z}(k) &= \frac{1}{k-\ell}\Delta_\ell \bar{w}(k), \\
\bar{\gamma}(k) &= \ell^{q-\frac{1}{2}}\left(\ell^{q}\frac{\sqrt{k+\ell}}{2k^{q+1}} + \frac{2}{k}\sqrt{k-\ell}\,\Delta_\ell\sqrt{k}\,\Delta_\ell\frac{1}{\sqrt{k-\ell}} + \frac{1}{k}\Delta_\ell^2\sqrt{k-\ell}\right)\bar{v}(k), \\
\bar{\beta}(k) &= \left(\frac{2(k-\ell)}{k\sqrt{k}}\right)\Delta_\ell\sqrt{k} - \frac{\ell}{k},
\end{aligned}$$

and applying similar arguments as in the previous theorem one can see that there exists a positive integer k_1 such that $u(k) = 0$ for all $k \in [k_1, \infty)$. □

In the next we present some formulae and examples to find the values of finite and infinite series in number theory as application of Δ_ℓ. First of all we need the following theorem.

Theorem 3.6 *Let $k \in [\ell, \infty)$ and $\ell \in (0, \infty)$. Then*

$$\sum_{r=1}^{[\frac{k}{\ell}]+s} \frac{(k - r\ell + 2\ell)^2 - 3\ell^2}{\ell^r (k - r\ell + 4\ell)_\ell^{(2)} (k - r\ell + \ell)_\ell^{(\lceil \frac{k-r\ell+\ell}{\ell} \rceil)}} = \frac{c_j}{\ell^{\lceil \frac{k}{\ell} \rceil}} - \frac{1}{(k + 3\ell) k_\ell^{(\lceil \frac{k}{\ell} \rceil)}}, \tag{40}$$

where $s = -1$ for $k \in \mathbb{N}_\ell(\ell)$, $s = 0$ for $k \notin \mathbb{N}_\ell(\ell)$ and each c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - [\frac{k}{\ell}]\ell$. In particular c_j is obtained from (40) by substituting $k = \ell + j$. Further

$$\sum_{r=0}^{\infty} \frac{(k + r\ell)^3 - \ell^3}{\ell^r ((k + r\ell)^2 - 2\ell^2)_\ell^{(2)} (k + r\ell + \ell)_\ell^{(\lceil \frac{k+r\ell+\ell}{\ell} \rceil)}} = \frac{1}{((k - \ell)^2 - 2\ell^2) k_\ell^{(\lceil \frac{k}{\ell} \rceil)}}. \tag{41}$$

Proof By Definition 2.1, we find

$$\Delta_\ell^{-1} \frac{((k + 2\ell)^2 - 3\ell^2)\ell^{\lceil \frac{k}{\ell} \rceil}}{(k + 4\ell)_\ell^{(2)} (k + \ell)_\ell^{(\lceil \frac{k+\ell}{\ell} \rceil)}} = c_j - \frac{\ell^{\lceil \frac{k}{\ell} \rceil}}{(k + 3\ell) k_\ell^{(\lceil \frac{k}{\ell} \rceil)}}$$

and (40) follows by Lemma 2.9 and

$$\frac{(k - ([\frac{k}{\ell}] + s)\ell + 2\ell)^2 - 3\ell^2}{(k - ([\frac{k}{\ell}] + s)\ell + 4\ell)_\ell^{(2)} (k - ([\frac{k}{\ell}] + s)\ell + \ell)_\ell^{(\lceil \frac{k - ([\frac{k}{\ell}] + s)\ell + \ell}{\ell} \rceil)}} \geq 0.$$

□

The following example illustrates Theorem 3.6.

Example 3.7 By taking $\ell = 1.7$, $k = 2$ and $j = 0.3$ in (40), we get $c_j = \frac{85}{81}$ and hence (40) becomes

$$\begin{aligned} &\sum_{r=1}^{[\frac{k}{\ell}]} \frac{(k - 1.7r + 2(1.7))^2 - 3(1.7)^2}{1.7^r (k - 1.7r + 4(1.7))_{1.7}^{(2)} (k - 1.7r + 1.7)_{1.7}^{(\lceil \frac{k-1.7r+1.7}{1.7} \rceil)}} \\ &\quad = \frac{85}{81(1.7)^{\lceil \frac{k}{1.7} \rceil}} - \frac{1}{(k + 3(1.7)) k_{1.7}^{(\lceil \frac{k}{1.7} \rceil)}}, \quad k = 2, 3.7, 5.4, \ldots \end{aligned}$$

Example 3.8 Taking $\ell = 3.5$ in (41), we obtain

$$\begin{aligned} &\sum_{r=0}^{\infty} \frac{(k + 3.5r)^3 - 3.5^3}{3.5^r ((k + 3.5r)^2 - 2(3.5)^2)_\ell^{(2)} (k + 3.5r + 3.5)_{3.5}^{(\lceil \frac{k+3.5r+3.5}{3.5} \rceil)}} \\ &\quad = \frac{1}{((k - 3.5)^2 - 2(3.5)^2) k_{3.5}^{(\lceil \frac{k}{3.5} \rceil)}}. \end{aligned}$$

In particular, when $k = 9$, above series becomes

$$\frac{9^3 - 3.5^3}{(9^2 - 2(3.5)^2)^{(2)}_{3.5} 12.5^{(4)}_{3.5}} + \frac{12.5^3 - 3.5^3}{3.5(12.5^2 - 2(3.5)^2)^{(2)}_{3.5} 16^{(5)}_{3.5}} + \frac{16^3 - 3.5^3}{3.5^2(16^2 - 2(3.5)^2)^{(2)}_{3.5} 19.5^{(6)}_{3.5}} + \cdots = \frac{1}{(5.5^2 - 2(3.5)^2)9^{(3)}_{3.5}}.$$

4 Concluding remarks

In the difference equations there are several interesting development, see for example, [4–6], and [8–16]. Recently, in [7], the fractional h-difference equations was studied. In the present work we study the $\ell_{2(\ell)}$ and $c_{0(\ell)}$ solutions of the second order generalized difference equation

$$\Delta^2_\ell u(k) + f\big(k, u(k)\big) = 0, \quad k \in [a, \infty), a > 0$$

and we prove the condition for non existence of non-trivial solution.

Competing interests
The authors declare that they do not have competing interest.

Authors' contributions
All the authors contributed equally.

Author details
[1]Department of Science and Humanities, R.M.D. Engineering College, Kavaraipettai, Tamil Nadu 601 206, India. [2]Department of Mathematics and Institute for Mathematical Research, University Putra Malaysia, Serdang, Selangor 43400, Malaysia. [3]Department of Mathematics, Sacred Heart College, Vellore District, Tirupattur, Tamil Nadu 635601, India.

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